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POWER SERIES METHODS III
- THE WAVE EQUATION

Robert D. Small

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ABSTRACT

The power series method used by the author to generate highly accurate finite difference schemes for ordinary differential equations in [1] and for the heat equation in [2] is here applied to the wave equation. The analysis runs parallel to [2] and involves semi-discrete approximations in t and in x before the totally discrete scheme is derived. The results differ from [2] in that an arbitrarily accurate difference scheme is found for the wave equation that is stable and consistent with the differential equation. No such scheme exists for the heat equation. The step sizes in x and t must be equal for this difference scheme. Other difference schemes that do not restrict the step sizes are stable only when the order of accuracy in x is less than 5. The lowest order scheme is shown to coincide with Keller's Box Scheme [3].

AMS (MOS) Subject Classifications: 35A40, 65M05, 65M10.

Key Words: Partial differential equation, wave equation, power series, difference scheme, high accuracy.

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SIGNIFICANCE AND EXPLANATION

This paper, a sequel to TSR #1923 and TSR #1924 where the author introduced the "Power Series Method" and applied it to the heat equation, applies the method to the wave equation. We derive difference equations that approximate the differential equation by substituting power series into it, and let these series apply in a grid of cells that cover the domain. After the coefficients in the series are evaluated the series are truncated at some order of accuracy. As much of the detail of the method is described in the above quoted TSR's, this paper omits such detail and moves rapidly through the analysis. The results are surprisingly different for the wave equation compared to those of the heat equation. Here we obtain a set of difference schemes of increasing accuracy that are stable to all orders while, for the heat equation, only schemes accurate to order 15 in x are stable.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

POWER SERIES METHODS III - THE WAVE EQUATION

Robert D. Small

Introduction

In previous papers of this series, [1], [2], which we refer to as I and II, the power series method has been used to derive finite difference schemes for a general first order ordinary differential equation and for the heat equation. This paper continues the study of the method by application to the wave equation. Since we shall depend heavily on the previous work, a I or II preceding equation or figure numbers will refer to equations or figures in these papers. In addition, since the treatment of the wave equation follows closely that of the heat equation, a good deal of detail in the procedures is omitted in this presentation. As in II, we discretize the wave equation first in t, then in x and finally combine the two discretizations, maintaining the stability conditions that have accumulated along the way. It turns out that one can obtain a stable, arbitrarily accurate difference approximation for the wave equation whereas this is impossible for the heat equation. As in the case of the heat equation the root locus diagram of equation (I-16) illustrated in Figure I-1 plays a fundamental role in the stability analysis.

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1. Formulation of the Problem

We now prepare to derive difference schemes for the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad . \tag{1}$$

The domain is taken identical to that used in II and is defined by $0 \le x \le 2Lh$, $0 \le t < \infty$. It is partitioned into a grid of rectangular cells of size $2h$ by $2k$ in the x and t directions respectively and each is identified by an ordered pair of integers (i,j). A local origin is located in each cell a distance $2\lambda k$ from the bottom of the cell on the vertical mid-line and the local co-ordinates are denoted (x_i,t_j), the solution to (1) then being denoted within the cell by $u_{ij}(x_i,t_j)$. As in II, since the preponderance of coefficients requires that subscripts be reserved to distinguish them, partial derivatives will always be written with the symbol ∂ and the superscript (n) will denote the nth ordinary derivative of a function with respect to its argument. Commas are used to separate subscript expressions and are omitted when no ambiquity arises.

We specify that initially u is $U(x)$ and $\frac{\partial u}{\partial t}$ is $V(x)$. For boundary conditions, u is $g_1(t)$ on the left side of the domain and $g_2(t)$ on the right. Expressing these conditions in local co-ordinates, we have

$$\left\{\begin{array}{ll} \left.\begin{array}{ll} u_{i1}(x_i,-2\lambda k) & = U_i(x_i) \\ \dfrac{\partial u_{i1}}{\partial t_j}(x_i,-2\lambda k) & = V_i(x_i) \end{array}\right\} & i = 1,2,\cdots,L \\ \\ \left.\begin{array}{ll} u_{1j}(-h,t_j) & = g_{1j}(t_j) \\ u_{Lj}(h,t_j) & = g_{2j}(t_j) \end{array}\right\} & j = 1,2,\cdots,\infty \end{array}\right. \quad . \tag{2}$$

For interior cell boundaries we specify that u and $\frac{\partial u}{\partial t}$ are continuous across horizontal boundaries and that u and $\frac{\partial u}{\partial x}$ are continuous across vertical boundaries. These conditions are

$$
(3)\quad
\begin{cases}
\left.\begin{aligned}
u_{ij}(x_i,-2\lambda k) &= u_{i,j-1}(x_i,2(1-\lambda)k) \\
\frac{\partial u_{ij}}{\partial t_j}(x_i,-2\lambda k) &= \frac{\partial u_{i,j-1}}{\partial t_{j-1}}(x_i,2(1-\lambda)k)
\end{aligned}\right\} & \begin{cases} i = 1,2,\cdots,L \\ j = 2,3,\cdots,\infty \end{cases} \\
\left.\begin{aligned}
u_{ij}(-h,t_j) &= u_{i-1,j}(h,t_j) \\
\frac{\partial u_{ij}}{\partial x_i}(-h,t_j) &= \frac{\partial u_{i-1,j}}{\partial x_{i-1}}(h,t_j)
\end{aligned}\right\} & \begin{cases} i = 2,3,\cdots,L \\ j = 1,2,\cdots,\infty \end{cases}
\end{cases}
.
$$

We now proceed with the derivation of truncated power series solutions to the equations, first with powers of t, then with powers of x and then with the combined series.

2. Discretization in t

Substitution of a power series in t into (1) gives the local solution

$$u_{ij}(x_i,t_j) = \sum_{n=0}^{\infty} \frac{a_{ij}^{(2n)}(x_i)}{(2n)!} t_j^{2n} + \sum_{n=0}^{\infty} \frac{b_{ij}^{(2n)}(x_i)}{(2n+1)!} t_j^{2n+1} \quad . \tag{4}$$

Substitution of this into (2) and (3) leads to the following infinite system of ordinary differential equations:

$$\text{(5a)} \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \frac{a_{i1}^{(2n)}(x_i)}{(2n)!} (-2\lambda k)^{2n} + \sum_{n=0}^{\infty} \frac{b_{i1}^{(2n)}(x_i)}{(2n+1)!} (-2\lambda k)^{2n+1} = U_i(x_i) \\ &\sum_{n=0}^{\infty} \frac{a_{i1}^{(2n+2)}(x_i)}{(2n+1)!} (-2\lambda k)^{2n+1} + \sum_{n=0}^{\infty} \frac{b_{i1}^{(2n)}(x_i)}{(2n)!} (-2\lambda k)^{2n} = V_i(x_i) \end{aligned} \right\} \; i = 1,2,\cdots,L$$

$$\text{(5b)} \quad \left\{ \begin{aligned} a_{1j}^{(2n)}(-h) &= g_{1j}^{(2n)}(0) \\ b_{1j}^{(2n)}(-h) &= g_{1j}^{(2n+1)}(0) \\ a_{Lj}^{(2n)}(h) &= g_{2j}^{(2n)}(0) \\ b_{Lj}^{(2n)}(h) &= g_{2j}^{(2n+1)}(0) \end{aligned} \right\} \left\{ \begin{aligned} n &= 0,1,\cdots,\infty \\ j &= 1,2,\cdots,\infty \end{aligned} \right.$$

$$\text{(5c)} \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \frac{a_{ij}^{(2n)}(x_i)}{(2n)!} (-2\lambda k)^{2n} + \sum_{n=0}^{\infty} \frac{b_{ij}^{(2n)}(x_i)}{(2n+1)!} (-2\lambda k)^{2n+1} = \\ &\qquad \sum_{n=0}^{\infty} \frac{a_{i,j-1}^{(2n)}(x_i)}{(2n)!} (2(1-\lambda)k)^{2n} + \sum_{n=0}^{\infty} \frac{b_{i,j-1}^{(2n)}(x_i)}{(2n+1)!} (2(1-\lambda)k)^{2n+1} \\ &\sum_{n=0}^{\infty} \frac{a_{ij}^{(2n+2)}(x_i)}{(2n+1)!} (-2\lambda k)^{2n+1} + \sum_{n=0}^{\infty} \frac{b_{ij}^{(2n)}(x_i)}{(2n)!} (-2\lambda k)^{2n} = \\ &\qquad \sum_{n=0}^{\infty} \frac{a_{i,j-1}^{(2n+2)}(x_i)}{(2n+1)!} (2(1-\lambda)k)^{2n+1} + \sum_{n=0}^{\infty} \frac{b_{i,j-1}^{(2n)}(x_i)}{(2n)!} (2(1-\lambda)k)^{2n} \end{aligned} \right\} \left\{ \begin{aligned} i &= 1,2,\cdots,L \\ j &= 2,3,\cdots,\infty \end{aligned} \right.$$

$$\text{(5d)} \quad \left\{ \begin{aligned} a_{ij}^{(n)}(-h) &= a_{i-1,j}^{(n)}(h) \\ b_{ij}^{(n)}(-h) &= b_{i-1,j}^{(n)}(h) \end{aligned} \right\} \left\{ \begin{aligned} n &= 0,1,\cdots,\infty \\ i &= 2,3,\cdots,L \\ j &= 1,2,\cdots,\infty \end{aligned} \right. \quad .$$

As is usual in the semi-discrete formulation, (5d) indicates that the partitioning into cells in the x-direction is unnecessary; the problem takes place in horizontal strips. Equations (5c) are truncated in the following manner.

$$\sum_{n=0}^{N_1} + \sum_{n=0}^{N_2} = \sum_{n=0}^{N_1} + \sum_{n=0}^{N_2}$$

$$\sum_{n=0}^{N_3} + \sum_{n=0}^{N_4} = \sum_{n=0}^{N_3} + \sum_{n=0}^{N_4} .$$

This system is tested for stability by the substitution

$$a_{ij}(x_i) = \beta^j e^{\hat{i}\alpha x_i} \hat{a}$$

$$b_{ij}(x_i) = \beta^j e^{\hat{i}\alpha x_i} \hat{b} ,$$

where $\hat{i} = \sqrt{-1}$.

The result is a system of homogeneous equations for which the condition for non-trivial solutions is

$$\sum_{n=0}^{N_1} \frac{(-1)^n[\lambda^{2n} - \beta^{-1}(1-\lambda)^{2n} \ (2\alpha k)^{2n}}{(2n)!} \sum_{n=0}^{N_4} \frac{(-1)^n[\lambda^{2n} - \beta^{-1}(1-\lambda)^{2n}](2\alpha k)^{2n}}{(2n)!} +$$

$$\sum_{n=0}^{N_2} \frac{(-1)^n[\lambda^{2n+1} + \beta^{-1}(1-\lambda)^{2n+1}](2\alpha k)^{2n+1}}{(2n+1)!} \sum_{n=0}^{N_3} \frac{(-1)^n[\lambda^{2n+1} + \beta^{-1}(1-\lambda)^{2n+1}](2\alpha k)^{2n+1}}{(2n+1)!} = 0 .$$

The choice $N_1 = N_4$ and $N_2 = N_3$ allows square roots to be taken and also suppresses some undesirable roots for β. The the best accuracy occurs for $N_1 = [N/2]$ and $N_2 = [\frac{N-1}{2}]$ for some order of accuracy N. Making these choices we solve for β obtaining

$$\beta = \frac{\displaystyle\sum_{n=0}^{N_1} \frac{(-1)^n(2(1-\lambda)\alpha k)^{2n}}{(2n)!} + \hat{i} \sum_{n=0}^{N_2} \frac{(-1)^n(2(1-\lambda)\alpha k)^{2n+1}}{(2n+1)!}}{\displaystyle\sum_{n=0}^{N_1} \frac{(-1)^n(2\lambda\alpha k)^{2n}}{(2n)!} + \hat{i} \sum_{n=0}^{N_2} \frac{(-1)^n(2\lambda\alpha k)^{2n+1}}{(2n+1)!}}$$

and its complex conjugate. We can obtain $|\beta| = 1$ by choosing $\lambda = 1/2$ but other values of λ do not consistently maintain $|\beta| \leq 1$ for all α or k, hence we take $\lambda = 1/2$. With

these choices we select the correct boundary conditions from (5b) and (5d) to suit the differential equations and obtain the following semi-discrete scheme that is still continuous in x.

$$\text{(6a)}\quad \left\{\begin{aligned} &\sum_{n=0}^{N_1} \frac{a_{i1}^{(2n)}(x_i)}{(2n)!}k^{2n} - \sum_{n=0}^{N_2} \frac{b_{i1}^{(2n)}(x_i)}{(2n+1)!}k^{2n+1} = U_i(x_i) \\ &-\sum_{n=0}^{N_2} \frac{a_{i1}^{(2n+2)}(x_i)}{(2n+1)!}k^{2n+1} + \sum_{n=0}^{N_1} \frac{b_{i1}^{(2n)}(x_i)}{(2n)!}k^{2n} = V_i(x_i) \end{aligned}\right\} i = 1,2,\cdots,L$$

$$\text{(6b)}\quad \left\{\begin{aligned} &a_{1j}^{(2n)}(-h) = g_{1j}^{(2n)}(0) && n = 0,1,\cdots,N_2 \\ &b_{1j}^{(2n)}(-h) = g_{1j}^{(2n+1)}(0) && n = 0,1,\cdots,N_1-1 \\ &a_{Lj}^{(2n)}(h) = g_{2j}^{(2n)}(0) && n = 0,1,\cdots,N_2 \\ &b_{Lj}^{(2n)}(h) = g_{2j}^{(2n+1)}(0) && n = 0,1,\cdots,N_1-1 \end{aligned}\right\} j = 1,2,\cdots,\infty$$

$$\text{(6c)}\quad \left\{\begin{aligned} &\sum_{n=0}^{N_1} \frac{a_{ij}^{(2n)}(x_i)}{(2n)!}k^{2n} - \sum_{n=0}^{N_2} \frac{b_{ij}^{(2n)}(x_i)}{(2n+1)!}k^{2n+1} = \\ &\qquad \sum_{n=0}^{N_1} \frac{a_{i,j-1}^{(2n)}(x_i)}{(2n)!}k^{2n} + \sum_{n=0}^{N_2} \frac{b_{i,j-1}^{(2n)}(x_i)}{(2n+1)!}k^{2n+1} \\ &-\sum_{n=0}^{N_2} \frac{a_{ij}^{(2n+2)}(x_i)}{(2n+1)!}k^{2n+1} + \sum_{n=0}^{N_1} \frac{b_{ij}^{(2n)}(x_i)}{(2n)!}k^{2n} = \\ &\qquad \sum_{n=0}^{N_2} \frac{a_{i,j-1}^{(2n+2)}(x_i)}{(2n+1)!}k^{2n+1} + \sum_{n=0}^{N_1} \frac{b_{i,j-1}^{(2n)}(x_i)}{(2n)!}k^{2n} \end{aligned}\right\} \left\{\begin{aligned} &i = 1,2,\cdots,L \\ &j = 2,3,\cdots,\infty \end{aligned}\right.$$

$$\text{(6d)}\quad \left\{\begin{aligned} &a_{ij}^{(n)}(-h) = a_{i-1,j}^{(n)}(h) && n = 0,1,\cdots,2N_2+1 \\ &b_{ij}^{(n)}(-h) = b_{i-1,j}^{(n)}(h) && n = 0,1,\cdots,2N_1-1 \end{aligned}\right\} \left\{\begin{aligned} &i = 2,3,\cdots,L \\ &j = 1,2,\cdots,\infty \end{aligned}\right.$$

$$\text{(6e)}\quad u_{ij}(x_i,t_j) = \sum_{n=0}^{N_1} \frac{a_{ij}^{(2n)}(x_i)}{(2n)!}t_j^{2n} + \sum_{n=0}^{N_2} \frac{b_{ij}^{(2n)}(x_i)}{(2n+1)!}t_j^{2n+1} \quad \left\{\begin{aligned} &i = 1,2,\cdots,L \\ &j = 1,2,\cdots,\infty \end{aligned}\right. .$$

3. Discretization in x

We begin with the power series in x_i which satisfies (1),

$$u_{ij}(x_i,t_j) = \sum_{m=0}^{\infty} \frac{\bar{a}_{ij}^{(2m)}(t_j)}{(2m)!} x_i^{2m} + \sum_{m=0}^{\infty} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m+1)!} x_i^{2m+1} \quad . \tag{7}$$

Substitution into (2) and (3) gives the following system, where we have taken $\lambda = 1/2$.

$$\left\{\begin{aligned} \bar{a}_{i1}^{(2m)}(-k) &= U_i^{(2m)}(0) \\ \bar{b}_{i1}^{(2m)}(-k) &= U_i^{(2m+1)}(0) \\ \bar{a}_{i1}^{(2m+1)}(-k) &= V_i^{(2m)}(0) \\ \bar{b}_{i1}^{(2m+1)}(-k) &= V_i^{(2m+1)}(0) \end{aligned}\right\} \left\{\begin{aligned} &m = 0,1,\cdots,\infty \\ &i = 1,2,\cdots,L \end{aligned}\right. \tag{8a}$$

$$\left\{\begin{aligned} \sum_{m=0}^{\infty} \frac{\bar{a}_{1j}^{(2m)}(t_j)}{(2m)!} h^{2m} - \sum_{m=0}^{\infty} \frac{\bar{b}_{1j}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} &= g_{1j}(t_j) \\ \sum_{m=0}^{\infty} \frac{\bar{a}_{Lj}^{(2m)}(t_j)}{(2m)!} h^{2m} + \sum_{m=0}^{\infty} \frac{\bar{b}_{Lj}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} &= g_{2j}(t_j) \end{aligned}\right\} \quad j = 1,2,\cdots,\infty \tag{8b}$$

$$\left\{\begin{aligned} \bar{a}_{ij}^{(m)}(-k) &= \bar{a}_{i,j-1}^{(m)}(k) \\ \bar{b}_{ij}^{(m)}(-k) &= \bar{b}_{i,j-1}^{(m)}(k) \end{aligned}\right\} \left\{\begin{aligned} &m = 0,1,\cdots,\infty \\ &i = 1,2,\cdots,L \\ &j = 2,3,\cdots,\infty \end{aligned}\right. \tag{8c}$$

$$\left\{\begin{aligned} &\sum_{m=0}^{\infty} \frac{\bar{a}_{ij}^{(2m)}(t_j)}{(2m)!} h^{2m} - \sum_{m=0}^{\infty} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} = \\ &\qquad \sum_{m=0}^{\infty} \frac{\bar{a}_{i-1,j}^{(2m)}(t_j)}{(2m)!} h^{2m} + \sum_{m=0}^{\infty} \frac{\bar{b}_{i-1,j}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} \\ &-\sum_{m=0}^{\infty} \frac{\bar{a}_{ij}^{(2m+2)}(t_j)}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{\infty} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m)!} h^{2m} = \\ &\qquad \sum_{m=0}^{\infty} \frac{\bar{a}_{i-1,j}^{(2m+2)}(t_j)}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{\infty} \frac{\bar{b}_{i-1,j}^{(2m)}(t_j)}{(2m)!} h^{2m} \end{aligned}\right\} \left\{\begin{aligned} &i = 2,3,\cdots,L \\ &j = 1,2,\cdots,\infty \end{aligned}\right. \tag{8d}$$

We truncate (8d) in the following manner,

$$\sum_{m=0}^{M_1} - \sum_{m=0}^{M_2} = \sum_{m=0}^{M_1} + \sum_{m=0}^{M_2}$$

$$- \sum_{m=0}^{M_3} + \sum_{m=0}^{M_4} = \sum_{m=0}^{M_3} + \sum_{m=0}^{M_4} \quad ,$$

and test the stability of this new system by the substitution

$$\bar{a}_{ij}(t_j) = e^{\beta t_j} e^{2\hat{i}\alpha h} \hat{a}$$

$$\bar{b}_{ij}(t_j) = e^{\beta t_j} e^{2\hat{i}\alpha h} \hat{b} \quad .$$

The resulting system of homogeneous equations has non-trivial solutions if the following equation holds.

$$\sum_{m=0}^{M_1} \frac{(\beta h)^{2m}}{(2m)!} \sum_{m=0}^{M_4} \frac{(\beta h)^{2m}}{(2m)!} = \left(\frac{1+e^{-2\hat{i}\alpha h}}{1-e^{-2\hat{i}\alpha h}}\right)^2 \sum_{m=0}^{M_2} \frac{(\beta h)^{2m+1}}{(2m+1)!} \sum_{m=0}^{M_3} \frac{(\beta h)^{2m+1}}{(2m+1)!} \quad .$$

To suppress undesirable roots in β we take $M_1 = M_4$ and $M_2 = M_3$ which allows square roots to be taken. Then we set $M_1 = [M/2]$ and $M_2 = [\frac{M-1}{2}]$ to obtain optimum accuracy for some order of accuracy M. The factor $\left(\frac{1+e^{-2\hat{i}\alpha h}}{1-e^{-2\hat{i}\alpha h}}\right)^2$ reduces to $-\cot^2 \alpha h$ which we set equal to $-c^2$. With these choices we obtain

$$(9) \qquad \sum_{m=0}^{M_1} \frac{(\beta h)^{2m}}{(2m)!} = \hat{i}c \sum_{m=0}^{M_2} \frac{(\beta h)^{2m+1}}{(2m+1)!} \quad .$$

This is equation (I-16) whose root loci for parameter c were originally displayed in Figure I-1. This diagram is symmetric about the real and imaginary axes and is later reproduced in Figure 1. For stability we must have $\mathrm{Re}\ \beta \leq 0$ for all α and hence all c. From Figure 1 we can see that there are loci that lie in the first and fourth quadrants for $M > 4$ and hence this semi-discrete scheme is stable only for $M \leq 4$. In this way the x-discretization suffers a severe restriction on the order of accuracy of difference schemes that may be used.

Now selecting suitable boundary conditions from (8a) and (8c) we obtain the following semi-discrete scheme, continuous in t.

$$(10a)\qquad \left\{\begin{array}{lll} \bar{a}_{i1}^{(2m)}(-k) = U_i^{(2m)}(0) & & m = 0,1,\cdots,M_2 \\ \bar{b}_{i1}^{(2m)}(-k) = U_i^{(2m+1)}(0) & & m = 0,1,\cdots,M_1-1 \\ \bar{a}_{i1}^{(2m+1)}(-k) = V_i^{(2m)}(0) & & m = 0,1,\cdots,M_2 \\ \bar{b}_{i1}^{(2m+1)}(-k) = V_i^{(2m+1)}(0) & & m = 0,1,\cdots,M_1-1 \end{array}\right\} i = 1,2,\cdots,L$$

$$(10b)\qquad \left\{\begin{array}{l} \sum_{m=0}^{M_1} \frac{\bar{a}_{1j}^{(2m)}(t_j)}{(2m)!} h^{2m} - \sum_{m=0}^{M_2} \frac{\bar{b}_{1j}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} = g_{1j}(t_j) \\ \sum_{m=0}^{M_1} \frac{\bar{a}_{Lj}^{(2m)}(t_j)}{(2m)!} h^{2m} + \sum_{m=0}^{M_2} \frac{\bar{b}_{Lj}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} = g_{2j}(t_j) \end{array}\right\} j = 1,2,\cdots,\infty$$

$$(10c)\qquad \left\{\begin{array}{ll} \bar{a}_{ij}^{(m)}(-k) = \bar{a}_{i,j-1}^{(m)}(k) & m = 0,1,\cdots,2M_2+1 \\ \bar{b}_{ij}^{(m)}(-k) = \bar{b}_{i,j-1}^{(m)}(k) & m = 0,1,\cdots,2M_1-1 \end{array}\right\} \left\{\begin{array}{l} i = 1,2,\cdots,L \\ j = 2,3,\cdots,\infty \end{array}\right.$$

$$(10d)\qquad \left\{\begin{array}{l} \sum_{m=0}^{M_1} \frac{\bar{a}_{ij}^{(2m)}(t_j)}{(2m)!} h^{2m} - \sum_{m=0}^{M_2} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} = \\ \qquad \sum_{m=0}^{M_1} \frac{\bar{a}_{i-1,j}^{(2m)}(t_j)}{(2m)!} h^{2m} + \sum_{m=0}^{M_2} \frac{\bar{b}_{i-1,j}^{(2m)}(t_j)}{(2m+1)!} h^{2m+1} \\ - \sum_{m=0}^{M_2} \frac{\bar{a}_{ij}^{(2m+2)}(t_j)}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{M_1} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m)!} h^{2m} = \\ \qquad \sum_{m=0}^{M_2} \frac{\bar{a}_{i-1,j}^{(2m+2)}(t_j)}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{M_1} \frac{\bar{b}_{i-1,j}^{(2m)}(t_j)}{(2m)!} h^{2m} \end{array}\right\} \left\{\begin{array}{l} i = 2,3,\cdots,L \\ j = 1,2,\cdots,\infty \end{array}\right.$$

$$(10e)\qquad u_{ij}(x_i,t_j) = \sum_{m=0}^{M_1} \frac{\bar{a}_{ij}^{(2m)}(t_j)}{(2m)!} x_i^{2m} + \sum_{m=0}^{M_2} \frac{\bar{b}_{ij}^{(2m)}(t_j)}{(2m+1)!} x_i^{2m+1} \quad \left\{\begin{array}{l} i = 1,2,\cdots,L \\ j = 1,2,\cdots,\infty \end{array}\right. .$$

4. Full Discretization

We now discretize the two semi-discrete schemes (6) and (10) with respect to the variables that remain. To discretize (6) we set

$$a_{ij}(x_i) = \sum_{m=0}^{\infty} \frac{c_{ijm}}{m!} x_i^m \,, \quad b_{ij}(x_i) = \sum_{m=0}^{\infty} \frac{d_{ijm}}{m!} x_i^m \,.$$

Substitution into (6) leads to the system

$$\text{(11a)} \quad \left\{ \begin{aligned} &\sum_{n=0}^{N_1} \frac{c_{i1,m+2n}}{(2n)!} k^{2n} - \sum_{n=0}^{N_2} \frac{d_{i1,m+2n}}{(2n+1)!} k^{2n+1} = U_i^{(m)}(0) \\ -&\sum_{n=0}^{N_2} \frac{c_{i1,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{d_{i1,m+2n}}{(2n)!} k^{2n} = V_i^{(m)}(0) \end{aligned} \right\} \left\{ \begin{aligned} m &= 0,1,\cdots,\infty \\ i &= 1,2,\cdots,L \end{aligned} \right.$$

$$\text{(11b)} \quad \left\{ \begin{aligned} &\sum_{m=0}^{\infty} \frac{c_{1j,m+2n}}{m!} (-h)^m = g_{1j}^{(2n)}(0) && n = 0,1,\cdots,N_2 \\ &\sum_{m=0}^{\infty} \frac{d_{1j,m+2n}}{m!} (-h)^m = g_{1j}^{(2n+1)}(0) && n = 0,1,\cdots,N_1-1 \\ &\sum_{m=0}^{\infty} \frac{c_{Lj,m+2n}}{m!} h^m = g_{2j}^{(2n)}(0) && n = 0,1,\cdots,N_2 \\ &\sum_{m=0}^{\infty} \frac{d_{Lj,m+2n}}{m!} h^m = g_{2j}^{(2n+1)}(0) && n = 0,1,\cdots,N_1-1 \end{aligned} \right\} j = 1,2,\cdots,\infty$$

$$\text{(11c)} \quad \left\{ \begin{aligned} &\sum_{n=0}^{N_1} \frac{c_{ij,m+2n}}{(2n)!} k^{2n} - \sum_{n=0}^{N_2} \frac{d_{ij,m+2n}}{(2n+1)!} k^{2n+1} = \\ &\qquad \sum_{n=0}^{N_1} \frac{c_{i,j-1,m+2n}}{(2n)!} k^{2n} + \sum_{n=0}^{N_2} \frac{d_{i,j-1,m+2n}}{(2n+1)!} k^{2n+1} \\ -&\sum_{n=0}^{N_2} \frac{c_{ij,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{d_{ij,m+2n}}{(2n)!} k^{2n} = \\ &\qquad \sum_{n=0}^{N_2} \frac{c_{i,j-1,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{d_{i,j-1,m+2n}}{(2n)!} k^{2n} \end{aligned} \right\} \left\{ \begin{aligned} m &= 0,1,\cdots,\infty \\ i &= 1,2,\cdots,L \\ j &= 2,3,\cdots,\infty \end{aligned} \right.$$

$$
\text{(11d)}\quad \left\{\begin{array}{ll}
\displaystyle\sum_{m=0}^{\infty} \frac{c_{ij,m+n}}{m!}(-h)^m = \sum_{m=0}^{\infty} \frac{c_{i-1,j,m+n}}{m!} h^m & n = 0,1,\cdots,2N_2+1 \\[2ex]
\displaystyle\sum_{m=0}^{\infty} \frac{d_{ij,m+n}}{m!}(-h)^m = \sum_{m=0}^{\infty} \frac{d_{i-1,j,m+n}}{m!} h^m & n = 0,1,\cdots,2N_1-1
\end{array}\right\} \left\{\begin{array}{l} i = 2,3,\cdots,L \\ j = 1,2,\cdots,\infty \end{array}\right.
$$

$$
\text{(11e)}\quad u_{ij}(x_i,t_j) = \sum_{m=0}^{\infty}\sum_{n=0}^{N_1} \frac{c_{ij,m+2n}}{m!(2n)!} x_i^m t_j^{2n} + \sum_{m=0}^{\infty}\sum_{n=0}^{N_2} \frac{d_{ij,m+2n}}{m!(2n+1)!} x_i^m t_j^{2n+1} \quad \left\{\begin{array}{l} i = 1,2,\cdots,L \\ j = 1,2,\cdots,\infty \end{array}\right. .
$$

To discretize (10) we set

$$
\bar{a}_{ij}(t_j) = \sum_{n=0}^{\infty} \frac{\bar{c}_{ijn}}{n!} t_j^n , \quad \bar{b}_{ij}(t_j) = \sum_{n=0}^{\infty} \frac{\bar{d}_{ijn}}{n!} t_j^n .
$$

Substitution into (10) gives the system

$$
\text{(12a)}\quad \left\{\begin{array}{lll}
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{c}_{i1,2m+n}}{n!}(-k)^n = U_i^{(2m)}(0) & m = 0,1,\cdots,M_2 \\[2ex]
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{d}_{i1,2m+n}}{n!}(-k)^n = U_i^{(2m+1)}(0) & m = 0,1,\cdots,M_1-1 \\[2ex]
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{c}_{i1,2m+n+1}}{n!}(-k)^n = V_i^{(2m)}(0) & m = 0,1,\cdots,M_2 \\[2ex]
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{d}_{i1,2m+n+1}}{n!}(-k)^n = V_i^{(2m+1)}(0) & m = 0,1,\cdots,M_1-1
\end{array}\right\} i = 1,2,\cdots,L
$$

$$
\text{(12b)}\quad \left\{\begin{array}{l}
\displaystyle\sum_{m=0}^{M_1} \frac{\bar{c}_{1j,2m+n}}{(2m)!} h^{2m} - \sum_{m=0}^{M_2} \frac{\bar{d}_{1j,2m+n}}{(2m+1)!} h^{2m+1} = g_{1j}^{(n)}(0) \\[2ex]
\displaystyle\sum_{m=0}^{M_1} \frac{\bar{c}_{Lj,2m+n}}{(2m)!} h^{2m} + \sum_{m=0}^{M_2} \frac{\bar{d}_{Lj,2m+n}}{(2m+1)!} h^{2m+1} = g_{2j}^{(n)}(0)
\end{array}\right\} \left\{\begin{array}{l} n = 0,1,\cdots,\infty \\ j = 1,2,\cdots,\infty \end{array}\right.
$$

$$
\text{(12c)}\quad \left\{\begin{array}{ll}
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{c}_{ij,m+n}}{n!}(-k)^n = \sum_{n=0}^{\infty} \frac{\bar{c}_{i,j-1,m+n}}{n!} k^n & m = 0,1,\cdots,2M_2+1 \\[2ex]
\displaystyle\sum_{n=0}^{\infty} \frac{\bar{d}_{ij,m+n}}{n!}(-k)^n = \sum_{n=0}^{\infty} \frac{\bar{d}_{i,j-1,m+n}}{n!} k^n & m = 0,1,\cdots,2M_1-1
\end{array}\right\} \left\{\begin{array}{l} i = 1,2,\cdots,L \\ j = 2,3,\cdots,\infty \end{array}\right.
$$

$$
(12d)\quad \left\{
\begin{aligned}
&\sum_{m=0}^{M_1} \frac{\bar{c}_{ij,2m+n}}{(2m)!} h^{2m} - \sum_{m=0}^{M_2} \frac{\bar{d}_{ij,2m+n}}{(2m+1)!} h^{2m+1} = \\
&\qquad \sum_{m=0}^{M_1} \frac{\bar{c}_{i-1,j,2m+n}}{(2m)!} h^{2m} + \sum_{m=0}^{M_2} \frac{\bar{d}_{i-1,j,2m+n}}{(2m+1)!} h^{2m+1} \\
&- \sum_{m=0}^{M_2} \frac{\bar{c}_{ij,2m+n+2}}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{M_1} \frac{\bar{d}_{ij,2m+n}}{(2m)!} h^{2m} = \\
&\qquad \sum_{m=0}^{M_2} \frac{\bar{c}_{i-1,j,2m+n+2}}{(2m+1)!} h^{2m+1} + \sum_{m=0}^{M_1} \frac{\bar{d}_{i-1,j,2m+n}}{(2m)!} h^{2m}
\end{aligned}
\right\}
\begin{cases} n = 0,1,\cdots,\infty \\ i = 2,3,\cdots,L \\ j = 1,2,\cdots,\infty \end{cases}
$$

$$
(12e)\quad u_{ij}(x_i,t_j) = \sum_{m=0}^{M_1} \sum_{n=0}^{\infty} \frac{\bar{c}_{ij,2m+n}}{(2m)!n!} x_i^{2m} t_j^n + \sum_{m=0}^{M_2} \sum_{n=0}^{\infty} \frac{\bar{d}_{ij,2m+n}}{(2m+1)!n!} x_i^{2m+1} t_j^n \begin{cases} i = 1,2,\cdots,L \\ j = 1,2,\cdots,\infty \end{cases} .
$$

Systems (11) and (12) express finite difference equations that are written in different variables and truncated at different stages of the series. To select common variables we compare (11e) and (12e) and note the correspondence

$$
\begin{aligned}
c_{ij,2n} &= \bar{c}_{ij,2n} \qquad & d_{ij,2n} &= \bar{c}_{ij,2n+1} \\
c_{ij,2n+1} &= \bar{d}_{ij,2n} \qquad & d_{ij,2n+1} &= \bar{d}_{ij,2n+1} .
\end{aligned}
$$

If (12) is now expressed in the variables of (11), we can choose all the finite limits of summation from this new system and from (11) and obtain the fully discrete scheme. That final system is

$$
(13a)\quad \left\{
\begin{aligned}
&\sum_{n=0}^{N_1} \frac{c_{i1,m+2n}}{(2n)!} k^{2n} - \sum_{n=0}^{N_2} \frac{d_{i1,m+2n}}{(2n+1)!} k^{2n+1} = U_i^{(m)}(0) \\
&- \sum_{n=0}^{N_2} \frac{c_{i1,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{d_{i1,m+2n}}{(2n)!} k^{2n} = V_i^{(m)}(0)
\end{aligned}
\right\}
\left\{ \begin{aligned} & m = 0,1,\cdots,M-1 \\ & i = 1,2,\cdots,L \end{aligned} \right.
$$

$$
\text{(13b)}\quad \left.\begin{cases}
\sum\limits_{m=0}^{M} \dfrac{c_{1j,m+2n}}{m!}(-h)^m = g_{1j}^{(2n)}(0) & n = 0,1,\cdots,N_2 \\[2ex]
\sum\limits_{m=0}^{M} \dfrac{d_{1j,m+2n}}{m!}(-h)^m = g_{1j}^{(2n+1)}(0) & n = 0,1,\cdots,N_1 - 1 \\[2ex]
\sum\limits_{m=0}^{M} \dfrac{c_{Lj,m+2n}}{m!}h^m = g_{2j}^{(2n)}(0) & n = 0,1,\cdots,N_2 \\[2ex]
\sum\limits_{m=0}^{M} \dfrac{d_{Lj,m+2n}}{m!}h^m = g_{2j}^{(2n+1)}(0) & n = 0,1,\cdots,N_1-1
\end{cases}\right\} j = 1,2,\cdots,\infty
$$

$$
\text{(13c)}\quad \left.\begin{cases}
\sum\limits_{n=0}^{N_1} \dfrac{c_{ij,m+2n}}{(2n)!}k^{2n} - \sum\limits_{n=0}^{N_2} \dfrac{d_{ij,m+2n}}{(2n+1)!}k^{2n+1} = \\[2ex]
\qquad \sum\limits_{n=0}^{N_1} \dfrac{c_{i,j-1,m+2n}}{(2n)!}k^{2n} + \sum\limits_{n=0}^{N_2} \dfrac{d_{i,j-1,m+2n}}{(2n+1)!}k^{2n+1} \\[2ex]
-\sum\limits_{n=0}^{N_2} \dfrac{c_{ij,m+2n+2}}{(2n+1)!}k^{2n+1} + \sum\limits_{n=0}^{N_1} \dfrac{d_{ij,m+2n}}{(2n)!}k^{2n} = \\[2ex]
\qquad \sum\limits_{n=0}^{N_2} \dfrac{c_{i,j-1,m+2n+2}}{(2n+1)!}k^{2n+1} + \sum\limits_{n=0}^{N_1} \dfrac{d_{i,j-1,m+2n}}{(2n)!}k^{2n}
\end{cases}\right\}\begin{cases} m = 0,1,\cdots,M-1 \\ i = 1,2,\cdots,L \\ j = 2,3,\cdots,\infty \end{cases}
$$

$$
\text{(13d)}\quad \left.\begin{cases}
\sum\limits_{m=0}^{M} \dfrac{c_{ij,m+n}}{m!}(-h)^m = \sum\limits_{m=0}^{M} \dfrac{c_{i-1,j,m+n}}{m!}h^m & n = 0,1,\cdots,2N_2+1 \\[2ex]
\sum\limits_{m=0}^{M} \dfrac{d_{ij,m+n}}{m!}(-h)^m = \sum\limits_{m=0}^{M} \dfrac{d_{i-1,j,m+n}}{m!}h^m & n = 0,1,\cdots,2N_1-1
\end{cases}\right\}\begin{cases} i = 2,3,\cdots,L \\ j = 1,2,\cdots,\infty \end{cases}
$$

$$
\text{(13e)}\quad u_{ij}(x_i,t_j) = \sum_{m=0}^{M}\sum_{n=0}^{N_1} \frac{c_{ij,m+2n}}{m!(2n)!}x_i^m t_j^{2n} + \sum_{m=0}^{M}\sum_{n=0}^{N_2} \frac{d_{ij,m+2n}}{m!(2n+1)!}x_i^m t_j^{2n+1}
$$

$$
- \left.\begin{cases}
\dfrac{c_{ij,M+N}}{M!N!}x_i^M t_j^N & (N \text{ even}) \\[2ex]
\dfrac{d_{ij,M+N-1}}{M!N!}x_i^M t_j^N & (N \text{ odd})
\end{cases}\right\}\begin{cases} i = 1,2,\cdots,L \\ j = 1,2,\cdots,\infty \end{cases}
$$

The final term in (13e) arises because the highest coefficients computed in (13a) to (13d) are $c_{ij,M+2N_2+1}$ and $d_{ij,M+2N_1-1}$. The double sum in (13e) contains one term that is not computed and this must be deleted.

We now test system (13) for stability. To examine one Fourier component of the error we set

$$c_{ijn} = \beta^j e^{2\hat{i}i\alpha h}\hat{c}_n$$

$$d_{ijn} = \beta^j e^{2\hat{i}i\alpha h}\hat{d}_n \quad .$$

Substitution of this into (13c) and (13d) gives the following set of homogeneous equations.

$$(14)\quad \begin{cases} \left.\begin{array}{l} \displaystyle\sum_{n=0}^{N_1} \frac{(1-\beta^{-1})}{(2n)!} k^{2n}\hat{c}_{m+2n} - \sum_{n=0}^{N_2} \frac{(1+\beta^{-1})}{(2n+1)!} k^{2n+1}\hat{d}_{m+2n} = 0 \\ \displaystyle -\sum_{n=0}^{N_2} \frac{(1+\beta^{-1})}{(2n+1)!} k^{2n+1}\hat{c}_{m+2n+2} + \sum_{n=0}^{N_1} \frac{(1-\beta^{-1})}{(2n)!} k^{2n}\hat{d}_{m+2n} = 0 \end{array}\right\} m = 0,1,\cdots,M-1 \\ \displaystyle\sum_{m=0}^{M} \frac{((-1)^m - e^{-2\hat{i}\alpha h})}{m!} h^m \hat{c}_{m+n} = 0 \qquad n = 0,1,\cdots,2N_2+1 \\ \displaystyle\sum_{m=0}^{M} \frac{((-1)^m - e^{-2\hat{i}\alpha h})}{m!} h^m \hat{d}_{m+n} = 0 \qquad n = 0,1,\cdots,2N_1-1 \quad . \end{cases}$$

The matrix of coefficients of (14) resembles the Sylvester matrix encountered in II except that this one has four sets of staggered rows instead of two. We note that if its determinant is set equal to zero, we obtain an algebraic equation of degree $2M$ in β and thus we seek $2M$ solutions for β for each fixed α. The substitution that collapses the equations is

$$\hat{c}_n = z^n, \qquad n = 0,1,\cdots,M+2N_2+1$$

$$\hat{d}_n = \pm z^{n+1}, \quad n = 0,1,\cdots,M+2N_1-1 \quad .$$

Then the two linearly independent equations that survive are

$$(15)\qquad \sum_{n=0}^{N_1} \frac{(kz)^{2n}}{(2n)!} = \pm \frac{1+\beta^{-1}}{1-\beta^{-1}} \sum_{n=0}^{N_2} \frac{(kz)^{2n+1}}{(2n+1)!}$$

and

$$(16) \qquad \sum_{m=0}^{M_1} \frac{(hz)^{2m}}{(2m)!} = \hat{i}c \sum_{m=0}^{M_2} \frac{(hz)^{2m+1}}{(2m+1)!}$$

where $c = \cot \alpha h$. *Equation (16) is of degree* M *in* z *and for fixed* c *gives* M *solutions.* Then equation (15), being linear in β but double valued, furnishes the full $2M$ solutions for β.

For non-trivial solutions for the errors we must have that equations (15) and (16) hold simultaneously. When $|\beta| = 1$ the factor $\frac{1+\beta^{-1}}{1-\beta^{-1}}$ is purely imaginary so that (15) is of the form (I-16). The root loci of this equation, displayed in Figure I-1, are reproduced here in Figure 1 for the variable kz. These curves divide the complex plane into regions where solutions of (15) fall for $|\beta| < 1$ and for $|\beta| > 1$. Since (15) is double-valued, however, any *point in Figure 1 that corresponds to a solution for* $|\beta| < 1$ *also corresponds to another* solution where $|\beta| > 1$. Thus any value of kz not on a curve or the imaginary axis of Figure 1 corresponds to an unstable error. Equation (16) is also of the form (I-16) and thus Figure 1 represents its root loci for the variable hz, parameterized by c.

For stability of any difference scheme (13) of order (M,N) we must have non-trivial errors only for $|\beta| \leq 1$. To investigate whether this is the case we identify the roots hz of (16) as the curves corresponding to M and the imaginary axis of Figure 1. These curves are mapped into the same diagram but for the scale adjusted to that of the variable kz so that the value of β *in (15) can be ascertained.* If the mapped curves fall only upon the curves corresponding to N and the imaginary axis then all errors occur for $|\beta| = 1$ and the scheme is stable. For any other case, some errors correspond to $|\beta| > 1$ and instability results.

Clearly scheme (13) is stable for all h, k, N if $M \leq 4$ since the solutions of (16) lie on the imaginary axis and, under the stretching of a scale change, remain there. This result we anticipated from section 3. An additional set of stable schemes is obtained by setting $M = N$ and $h = k$ since the roots of (15) for $|\beta| = 1$ and the roots of (16) then coincide. In this way we obtain stable difference schemes for (1) of arbitrarily high order.

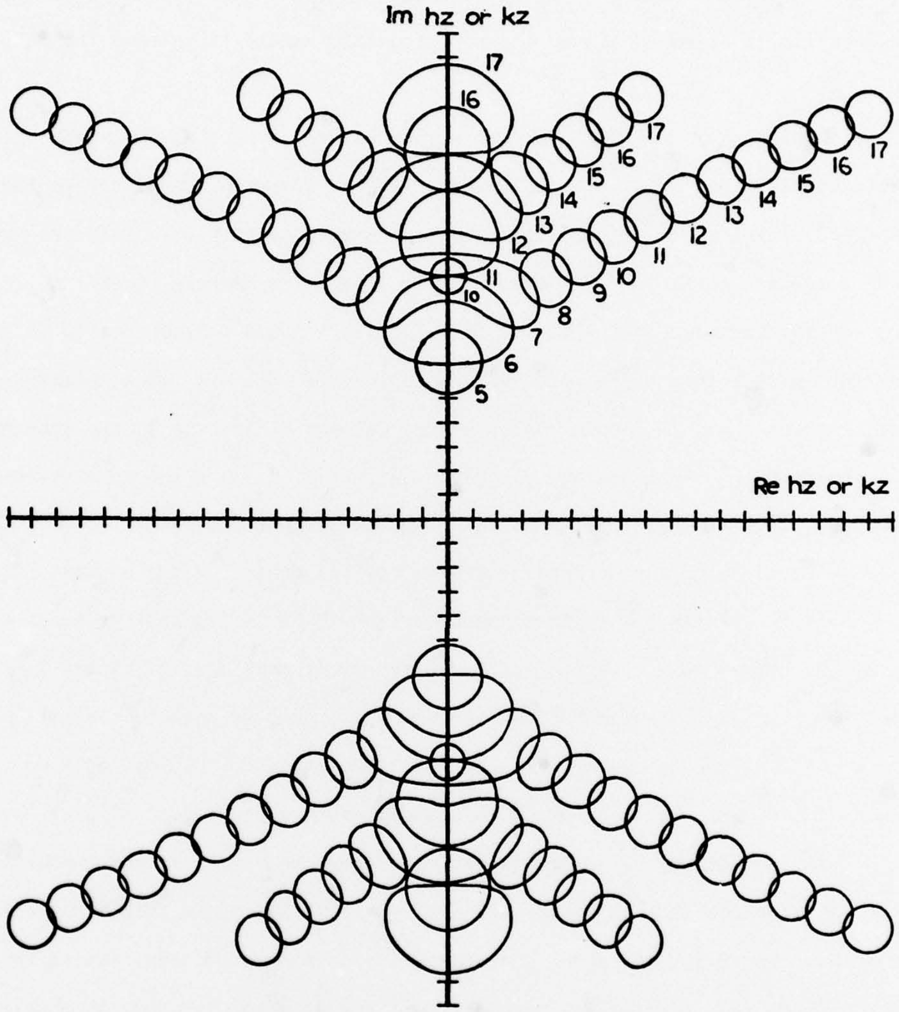

Figure 1: Stability Curves

These curves are root loci of equations (15) and (16). Numerals refer to the value of N or M.

5. Truncation Error and Consistency

We define the truncation error of a computed quantity as the difference between the computed and the exact solutions for it. Thus the truncation errors $\tilde{c}_{ijn}$, $\tilde{d}_{ijn}$ and $\tilde{u}_{ij}$ are given by

$$\tilde{c}_{ijn} = c_{ijn} - c_{ijn}^{(e)}$$

$$\tilde{d}_{ijn} = d_{ijn} - d_{ijn}^{(e)}$$

$$\tilde{u}_{ij} = u_{ij} - u_{ij}^{(e)}$$

where c_{ijn}, d_{ijn} and u_{ij} are the computed solutions and satisfy (13) for finite M and N while $c_{ijn}^{(e)}$, $d_{ijn}^{(e)}$ and $u_{ij}^{(e)}$ satisfy (13) for $M = \infty$, $N = \infty$. Substracting the two forms of (13) we obtain the following equations in the errors:

$$\text{(17a)} \left\{ \begin{aligned} &\sum_{n=0}^{N_1} \frac{\tilde{c}_{i1,m+2n}}{(2n)!} k^{2n} - \sum_{n=0}^{N_2} \frac{\tilde{d}_{i1,m+2n}}{(2n+1)!} k^{2n+1} = \\ &\qquad \sum_{n=N_1+1}^{\infty} \frac{c^{(e)}_{i1,m+2n}}{(2n)!} k^{2n} - \sum_{n=N_2+1}^{\infty} \frac{d^{(e)}_{i1,m+2n}}{(2n+1)!} k^{2n+1} \\ &-\sum_{n=0}^{N_2} \frac{\tilde{c}_{i1,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{\tilde{d}_{i1,m+2n}}{(2n)!} k^{2n} = \\ &\qquad -\sum_{n=N_2+1}^{\infty} \frac{c^{(e)}_{i1,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=N_1+1}^{\infty} \frac{d^{(e)}_{i1,m+2n}}{(2n)!} k^{2n} \end{aligned} \right\} \left\{ \begin{aligned} &m = 0,1,\cdots,M-1 \\ &i = 1,2,\cdots,L \end{aligned} \right.$$

$$\text{(17b)} \left\{ \begin{aligned} &\left. \begin{aligned} &\sum_{m=0}^{M} \frac{\tilde{c}_{ij,m+2n}}{m!} (-h)^m = \sum_{m=M+1}^{\infty} \frac{c^{(e)}_{ij,m+2n}}{m!} (-h)^m \\ &\sum_{m=0}^{M} \frac{\tilde{c}_{Lj,m+2n}}{m!} h^m = \sum_{m=M+1}^{\infty} \frac{c^{(e)}_{Lj,m+2n}}{m!} h^m \end{aligned} \right\} n = 0,1,\cdots,N_2 \\ &\left. \begin{aligned} &\sum_{m=0}^{M} \frac{\tilde{d}_{1j,m+2n}}{m!} (-h)^m = \sum_{m=M+1}^{\infty} \frac{d^{(e)}_{1j,m+2n}}{m!} (-h)^m \\ &\sum_{m=0}^{M} \frac{\tilde{d}_{Lj,m+2n}}{m!} h^m = \sum_{m=M+1}^{\infty} \frac{d^{(e)}_{Lj,m+2n}}{m!} h^m \end{aligned} \right\} n = 0,1,\cdots,N_1-1 \end{aligned} \right\} j = 1,2,\cdots,\infty$$

(17c)
$$\left\{\begin{aligned}
&\sum_{n=0}^{N_1} \frac{\tilde{c}_{ij,m+2n}-\tilde{c}_{i,j-1,m+2n}}{(2n)!}k^{2n} - \sum_{n=0}^{N_2}\frac{\tilde{d}_{ij,m+2n}+\tilde{d}_{i,j-1,m+2n}}{(2n+1)!}k^{2n+1} = \\
&\sum_{n=N_1+1}^{\infty} \frac{c^{(e)}_{ij,m+2n}-c^{(e)}_{i,j-1,m+2n}}{(2n)!}k^{2n} - \sum_{n=N_2+1}^{\infty}\frac{d^{(e)}_{ij,m+2n}+d^{(e)}_{i,j-1,m+2n}}{(2n+1)!}k^{2n+1} \\
&-\sum_{n=0}^{N_2} \frac{\tilde{c}_{ij,m+2n+2}+\tilde{c}_{i,j-1,m+2n+2}}{(2n+1)!}k^{2n+1} + \sum_{n=0}^{N_1}\frac{\tilde{d}_{ij,m+2n}-\tilde{d}_{i,j-1,m+2n}}{(2n)!}k^{2n} = \\
&-\sum_{n=N_2+1}^{\infty} \frac{c^{(e)}_{ij,m+2n+2}+c^{(e)}_{i,j-1,m+2n+2}}{(2n+1)!}k^{2n+1} + \sum_{n=N_1+1}^{\infty}\frac{d^{(e)}_{ij,m+2n}-d^{(e)}_{i,j-1,m+2n}}{(2n)!}k^{2n}
\end{aligned}\right\}
\left\{\begin{aligned} m &= 0,1,\cdots,M-1 \\ i &= 1,2,\cdots,L \\ j &= \quad 2,\cdots,\infty \end{aligned}\right.$$

(17d)
$$\left\{\begin{aligned}
&\sum_{m=0}^{M}\frac{\tilde{c}_{ij,m+n}(-h)^m-\tilde{c}_{i-1,j,m+n}h^m}{m!} = \sum_{m=M+1}^{\infty}\frac{c^{(e)}_{ij,m+n}(-h)^m-c^{(e)}_{i-1,j,m+n}h^m}{m!} \qquad n = 0,1,\cdots,2N_2+1 \\
&\sum_{m=0}^{M}\frac{\tilde{d}_{ij,m+n}(-h)^m-\tilde{d}_{i-1,j,m+n}h^m}{m!} = \sum_{m=M+1}^{\infty}\frac{d^{(e)}_{ij,m+n}(-h)^m-d^{(e)}_{i-1,j,m+n}h^m}{m!} \qquad n = 0,1,\cdots,2N_1-1
\end{aligned}\right\}$$

$$\left\{\begin{aligned} i &= 2,\cdots,L \\ j &= 1,2,\cdots,\infty \end{aligned}\right.$$

(17e)
$$\tilde{u}_{ij}(x_i,t_j) = \sum_{n=0}^{N_1}\sum_{m=0}^{M}\frac{\tilde{c}_{ij,m+2n}+c^{(e)}_{ij,m+2n}}{m!(2n)!}x_i^m t_j^{2n} +$$

$$\sum_{n=0}^{N_2}\sum_{m=0}^{M}\frac{\tilde{d}_{ij,m+2n}+d^{(e)}_{ij,m+2n}}{m!(2n+1)!}x_i^m t_j^{2n+1} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{c^{(e)}_{ij,m+2n}}{m!(2n)!}x_i^m t_j^{2n}$$

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{d^{(e)}_{ij,m+2n}}{m!(2n+1)!}x_i^m t_j^{2n+1} - \left\{\begin{aligned} &\frac{\tilde{c}_{ij,M+N}+c^{(e)}_{ij,M+N}}{M!N!}x_i^M t_j^N \qquad (N\ \text{even}) \\ &\frac{\tilde{d}_{ij,M+N-1}+d^{(e)}_{ij,M+N-1}}{M!N!}x_i^M t_j^N \qquad (N\ \text{odd}) \end{aligned}\right. .$$

$$i = 1,2,\cdots,L$$
$$j = 1,2,\cdots,\infty$$

If the determinant of the coefficient matrix of (17a) to (17d) is not zero then the solutions for the errors $\tilde{c}_{ijn}$ and $\tilde{d}_{ijn}$ are $O(h^{M+1},k^{N+1})$. If this is so then it follows from (17e) that $\tilde{u}_{ij}$ is $O(h^{M+1},k^{N+1},h^M k^N)$. The difference scheme (13) is consistent with the differential equation (1), the boundary conditions (2) and the continuity conditions (3) since the errors vanish as h and k approach zero, for all $M \geq 1$, $N \geq 1$, provided that the coefficient matrix of (17) is of full rank.

This last condition is investigated by examining the homogeneous system formed by deleting the terms on the right side of (17) and the errors at time step $j-1$. Then we may delete the subscript j obtaining the system

(18a)
$$\left\{\begin{aligned} &\sum_{n=0}^{N_1} \frac{\tilde{c}_{i,m+2n}}{(2n)!} k^{2n} - \sum_{n=0}^{N_2} \frac{\tilde{d}_{i,m+2n}}{(2n+1)!} k^{2n+1} = 0 \\ -&\sum_{n=0}^{N_2} \frac{\tilde{c}_{i,m+2n+2}}{(2n+1)!} k^{2n+1} + \sum_{n=0}^{N_1} \frac{\tilde{d}_{i,m+2n}}{(2n)!} k^{2n} = 0 \end{aligned}\right\} \left\{\begin{aligned} &m = 0,1,\cdots,M-1 \\ &i = 1,2,\cdots,L \end{aligned}\right.$$

(18b)
$$\left\{\begin{aligned} &\sum_{m=0}^{M} \frac{\tilde{c}_{1,m+2n}}{m!}(-h)^m = 0, \quad \sum_{m=0}^{M} \frac{\tilde{c}_{L,m+2n}}{m!} h^m = 0 \qquad n = 0,1,\cdots,N_2 \\ &\sum_{m=0}^{M} \frac{\tilde{d}_{1,m+2n}}{m!}(-h)^m = 0, \quad \sum_{m=0}^{M} \frac{\tilde{d}_{L,m+2n}}{m!} h^m = 0 \qquad n = 0,1,\cdots,N_1-1 \\ &\left.\begin{aligned} &\sum_{m=0}^{M} \frac{\tilde{c}_{i,m+n}(-h)^m - \tilde{c}_{i-1,m+n} h^m}{m!} = 0 \qquad n = 0,1,\cdots,2N_2+1 \\ &\sum_{m=0}^{M} \frac{\tilde{d}_{i,m+n}(-h)^m - \tilde{d}_{i-1,m+n} h^m}{m!} = 0 \qquad n = 0,1,\cdots,2N_1-1 \end{aligned}\right\} i = 2,3,\cdots,L \end{aligned}\right. .$$

The following substitution, motivated by the treatment of (14), turns out to give general solutions for (18):

(19)
$$\left\{\begin{aligned} \tilde{c}_{i,2n} &= A_i z^{2n} \\ \tilde{c}_{i,2n+1} &= B_i z^{2n+1} \\ \tilde{d}_{i,2n} &= C_i z^{2n} \\ \tilde{d}_{i,2n+1} &= D_i z^{2n+1} \end{aligned}\right. .$$

On substitution of (19) into (18), the following notation becomes convenient.

$$(20) \quad \begin{cases} H_1(h,z) = \sum_{m=0}^{M_1} \dfrac{(hz)^{2m}}{(2m)!} , & H_2(h,z) = \sum_{m=0}^{M_2} \dfrac{(hz)^{2m+1}}{(2m+1)!} \\ K_1(k,z) = \sum_{n=0}^{N_1} \dfrac{(kz)^{2n}}{(2n)!} , & K_2(k,z) = \sum_{n=0}^{N_2} \dfrac{(kz)^{2n+1}}{(2n+1)!} . \end{cases}$$

Then substitution of (19) into (18a) results in a number of equations that differ by factors that are powers of z. The linearly independent equations that survive are

$$\left.\begin{aligned} \begin{pmatrix} K_1 & -K_2 \\ -K_2 & K_1 \end{pmatrix} \begin{pmatrix} A_i \\ z^{-1}C_i \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \begin{pmatrix} K_1 & -K_2 \\ -K_2 & K_1 \end{pmatrix} \begin{pmatrix} B_i \\ z^{-1}D_i \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \end{aligned}\right\} i = 1,2,\cdots,L$$

For non-trivial solutions we require

$$(21) \qquad K_1(k,z) = \pm K_2(k,z)$$

and then we have

$$(22) \qquad \left.\begin{aligned} C_i &= \pm zA_i \\ D_i &= \pm zB_i \end{aligned}\right\} i = 1,2,\cdots,L .$$

Placing (19) into (18b) and using (20) and (22) we again obtain a large set of equations that reduces due to factoring to the independent set

$$(23) \quad \begin{cases} H_1A_1 - H_2B_1 = 0 \\ \left.\begin{aligned} H_1(A_i - A_{i-1}) - H_2(B_i + B_{i-1}) &= 0 \\ H_2(A_i + A_{i-1}) - H_1(B_i - B_{i-1}) &= 0 \end{aligned}\right\} i = 2,3,\cdots,L \\ H_1A_L + H_2B_L = 0 . \end{cases}$$

For (13) to be consistent with (1), (2) and (3) we must not have non-trivial solutions for the errors in (18). These errors are non-trivial if (21) is satisfied and, in addition, there

are non-trivial solutions to (23). To test for consistency for a given choice of M, N, L, h, k, we place k into (21) to determine z and then if the determinant of the coefficient matrix of (23) is non-zero, (13) is consistent. It would seem unlikely that a random choice of h and k would give an inconsistent scheme. We found in section 4 that to obtain a stable scheme for $M \geq 5$ we were forced to take $h = k$ and $M = N$. This choice happens to give a consistent scheme since (21) leads to $H_1 = \pm H_2$ and then the rows of the matrix in (23) are mutually orthogonal which implies that only trivial solutions exist. Thus we have a stable, consistent, and arbitrarily accurate difference approximation for (1).

As in II, it is difficult to prove that substitution (19) yields all solutions for (18) since one does not have a general formula for the determinant of the matrix in (23). The working of several examples, however, indicates that (19) is sufficiently general.

6. Lowest Order Scheme and Concluding Remarks

We have indicated in II that the lowest order scheme generated by the Power Series Method for the heat equation coincided with Keller's Box Scheme [3] for the heat equation. Making the same comparison here we set $M = 1$, $N = 1$ in (13) and eliminate the coefficients c_{ijn} and d_{ijn} in favor of the solution values u_{ij}. We obtain the following difference scheme, arranged to display the approximation of the derivatives of (1).

$$\frac{(u_{i-1,j+1}-2u_{i-1,j}+u_{i-1,j-1}) + 2(u_{i,j+1}-2u_{ij}+u_{i,j-1}) + (u_{i+1,j+1}-2u_{i+1,j}+u_{i+1,j-1})}{4(2k)^2}$$

$$= \frac{(u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}) + 2(u_{i-1,j}-2u_{i,j}+u_{i+1,j}) + (u_{i-1,j-1}-2u_{i,j-1}+u_{i+1,j-1})}{4(2h)^2} \quad .$$

Alternatively, if an auxilliary variable v is introduced into (1) so that we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial t}, \quad \frac{\partial u}{\partial t} = \frac{\partial v}{\partial x} \quad ,$$

then the Box Scheme [3] used on the derivatives of this system, with the resulting variables v eliminated, gives the above difference scheme. This demonstration strengthens our speculation made in II that the lowest order scheme in the Power Series Method applied to any differential equation gives the same result as Keller's Box Scheme when the latter can be applied.

In conclusion, having applied the Power Series Method to the wave equation, arbitrarily accurate difference schemes have been produced. The point of expansion of the power series within each cell was taken as the center of the cell in order that the t-discretization be stable. The heat equation at this stage required only that this point be at least as high as the center point and was later chosen to be the center point to make the total discretization stable. We speculate that the center point of the cell should always be the point of expansion of the series. The stability tests of sections 2 and 3 indicated that the t-discretization was unconditionally stable and that the x-discretization was stable only for $M \leq 4$. The stability test for the combine discretization corroborated these results in that $M \leq 4$ was unconditionally stable for any N. Another stable set of difference schemes that was not predicted by the semi-discrete stability tests was the case where $h = k$ and $M = N$. This is the set of

schemes of arbitrarily high order of accuracy, and there was no equivalent for the heat equation. In order for any difference scheme to be consistent with the differential equation the step sizes and orders of accuracy must meet a mild restriction. When $M \leq 4$, stable schemes must be tested for this restriction but the case $h = k$, $M = N$ was shown to satisfy the restriction.

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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The power series method used ~~by the author~~ to generate highly accurate finite difference schemes for ordinary differential equations in [1] and for the heat equation in [2] is here applied to the wave equation. The analysis runs parallel to [2] and involves semi-discrete approximations in t and in x before the totally discrete scheme is derived. The results differ ~~from [2]~~ in that an arbitrarily accurate difference scheme is found for the wave equation that is stable and consistent with the differential equation. No such scheme exists for the heat equation. The step sizes in x and t must be equal for this

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ABSTRACT (continued)

difference scheme. Other difference schemes that do not restrict the step sizes are stable only when the order of accuracy in x is less than 5. The lowest order scheme is shown to coincide with Keller's Box Scheme [3].